Figurate Numbers: A Historical Survey of an Ancient Mathematics

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Introduction

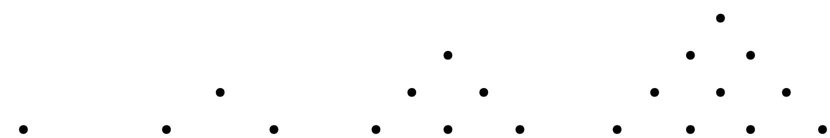
Figurate numbers comprise one of the oldest areas of mathematics, dating back to the Pythagoreans of the 6^{th} century BCE and capturing the attention of many mathematical luminaries, such as Fermat, Euler, and Gauss. In contemporary times, however, figurate numbers are studied only by student mathematicians, largely because of their aesthetic link to geometric objects. Interest in figurate numbers and in number theory generally has been in decline since the Age of Enlightenment, due to mathematicians' lean towards application and the scientific customer. Modern mathematics is largely defined by logic and scientific application rather than philosophical excursion or theoretical notion. Nonetheless, despite the professionalization of the subject of mathematics, figurate numbers remain a rich subject for teaching about philosophical relationships between arithmetic and geometry, and, while the body of knowledge still does not have any apparent application, perhaps future mathematics may find a diamond that has been lying in the rough for several millenia.

Origins

The earliest recorded forms of mathematics contain rudimentary concepts of numbers and counting with hardly any notion of abstract throught. The small archaeological record that does exist establishes the ancients' preoccupation with problems largely related to infrastructure and agriculture. Exercises in trade and commerce that appear on several archaic clay tablets and papyrus suggest that numbers were largely understood by the ancients as a property of the actual physical object(s). For example, the sensation of water may be described as hot, tepid, or cold, in relation to the internal temperature of the water. Regardless of whether one tests the temperature of water, the water will always have a property that describes its current internal temperature. Similarly for the ancient view of numbers, the abstract quantity of three is something that cannot be transferred among objects, such as from goats to coins: these numbers are different types of threes that belong to specific groups of objects. Ancient thinkers eventually discovered how to apply the abstract in mathematics, but the process was slow and chaotic. The first and perhaps largest step toward a modern understanding of numbers comes from the ancient cult of the Pythagoreans. Not much is known about the leader, Pythagoras, and what is known comes from historians who lived hundreds of years after Pythagoras's death. Despite the lack of contemporaneous reports, the records from Iamblicus, Aristotle, and others relay the life and philosophy of Pythagoras in the form of ancient biography.[1]

The mathematician Morris Kline highlights the interesting fact that the Pythagoreans did not completely develop an abstract notion of numbers, which is to say that, when the Pythagoreans claimed everything was made of numbers, they meant it in a literal sense. Similar to our current understanding that atoms are the building blocks of the universe, the Pythagoreans' belief was that numbers were our atoms.[3] The Pythagoreans did not believe that atoms were in the shape of numbers, but rather that each and every possible arrangement was a specific number. The Pythagoreans organized the various numbers first by geometric arrangement and then by size; today, mathematicians call these arithmetic progressions "figurate numbers." Using geometry as the basis for all physical things, the smallest geometric figure in the traditional sense is the triangle; hence, the base progression of figurate numbers is known as triangular numbers. Notations for figurate numbers vary, so this author has chosen a notation that lends itself well to the different aspects of the discussion. Let $F_m^d(n) = x$ denote a figurate number in dimension d with geometric arrangement m, side length count n and total unit count x, where $d, m, n, x \in \mathbb{N}$ and $m \geq 3$. Because the Pythagoreans held that the monad (the number one) was the most fundamental element from which all else stemmed, each geometric arrangement was composed of this self-replicated unit, as illustrated in Figure 2:

Figure 1: A depiction of Pythagoras of Samos[2]



$F_3^2(1) = 1$ $\quad$ $F_3^2(2) = 3$ $\quad$ $F_3^2(3) = 6$ $\quad$ $F_3^2(4) = 10$

Figure 2: The four smallest triangular numbers

If one considers the total count of units for each figure in Figure 2, this arithmetic series takes shape: $F_3^2(n) = \{1, 3, 6, 10, 15, 21, \ldots, \frac{n(n+1)}{2} : n \in \mathbb{N}\}$. Taking note of the difference between each triangle and the next, one can deduce that the next largest triangle will be formed by adding a row of $n + 1$ units to one of the three sides of the preceding figure. This deduction leads to the following formula for the total number of units in a triangular number:

$$\begin{aligned} F_3^2(n) &= F_3^2(n-1) + n \\ &= F_3^2(n-2) + (n-1) + n \\ &\;\;\vdots \\ &= 1 + 2 + 3 + \ldots + (n-1) + n = \sum_{i=1}^{n} i. \end{aligned}$$

Although it lacks the rigorous standards of modern proofs, the famous derivation of the triangular formula by the child Friedrich Gauss, later a noted mathematician, earns homage here:

$$\begin{array}{rl} F_3^2(n) = & 1 + \quad 2 + \quad 3 + \ldots + (n-1) + n \\ +F_3^2(n) = & n + (n-1) + (n-2) + \ldots + \quad 2 \;\; + 1 \end{array}$$

$$2F_3^2(n) = (n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1) = n(n+1)$$

$$\therefore F_3^2(n) = \frac{n(n+1)}{2}.$$

An absence of evidence suggests the Pythagoreans did not develop a concept of zero; hence, the smallest possible value for each arithmetic series is the number one. This idea—that one is the smallest possible value—is a central component of Pythagorean philosophy, which interestingly establishes a formulation for monotheism: every arithmetic progression of figurate numbers includes the monad, and the monad is the only number that appears in every geometric arrangement. Thus, to the Pythagoreans, the monad is a perfect, self-replicating unit that births every possible geometric shape: each figurate number is comprised of just the monad and self-replicated copies, which form the various sizes and arrangments. Perhaps this line of reasoning also explains why the Pythagoreans adopted the $F_3^2(4)$ triangle as their sacred object (triangular numbers serve as a basis for all other figurate arrangments) and also why they were devastated after the discovery of an incommensurate number: the foundation for their entire philosophy was disproven.

Theorem 1: Every integer greater than one can be expressed as the difference of two consecutive triangular numbers.

Proof: Since every triangular number can be expressed in the algebraic form $\frac{n(n+1)}{2}$*, the preceding triangular number may be expressed as* $\frac{(n-1)(n)}{2}$ *by replacing* n *with* $n-1$*, where* $n \in \mathbb{N}$*. Then, the application of algebraic laws provides the proof:*

$$\frac{n(n+1)}{2} - \frac{(n-1)(n)}{2} = \frac{1}{2}[n(n+1) - (n-1)(n)]$$

$$= \frac{1}{2}[(n^2+n) - (n^2-n)]$$

$$= \frac{1}{2}(n^2+n-n^2+n)$$

$$= \frac{1}{2}(2n)$$

$$= n.$$

The second arithmetic progression aligns itself to the geometric shape with four equilateral sides and congruent angles (see Figure 3), hence the name square numbers. While figurate numbers have been all but forgotten in the realm of mathematics, the naming convention and usage of square numbers are perhaps the only remnant still in use. In large part, mathematicians promoted square numbers over the others due to their ease of use and significance in quadratic equations.

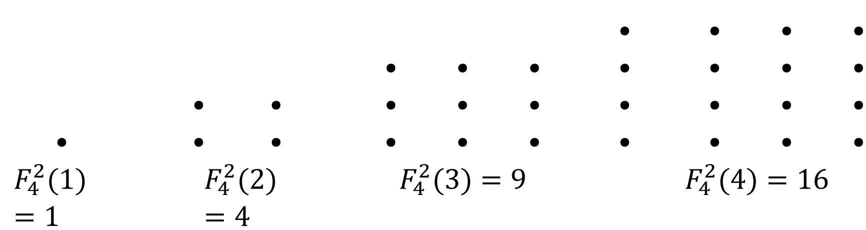


Figure 3: The four smallest square numbers

Using the same argument for the construction of triangular numbers, each square number is generated by adding a corner section along the top and right of the previous shape. This new strip that is repeatedly added to the previous figure is called a *gnomon,* and it is unique for each figurate number type. For square numbers, the gnomon takes on an "L" shape and matches the Greek word **γνωμων** (gnōmōn), the name for the L-shaped carpenter's square. The formulation of the arithmetic series of square numbers is $F_4^2(n) = \{1, 4, 9, 16, \ldots, n^2 : n \in \mathbb{N}\}$ and the difference between each number is simply each odd integer of corresponding size. Thus, one can deduce that the series for square numbers is

$$
\begin{aligned}
F_4^2(n) &= F_4^2(n-1) + (2n-1) \\
&= F_4^2(n-2) + (2n-1) + (2n-3) \\
&\vdots \\
&= 1 + 3 + 5 + \ldots + (2n-3) + (2n-1) = \textstyle\sum_{i=1}^{n} 2i - 1
\end{aligned}
$$

and can determine the formula for square numbers by applying the aforementioned Gaussian method:

$$
\begin{array}{rcccccccc}
F_4^2(n) = & 1 & + & 3 & + & 5 & + \ldots + & (2n-3) & + (2n-1) \\
+F_4^2(n) = & (2n-1) & + & (2n-3) & + & (2n-5) & + \ldots + & 3 & + \quad 1 \\
\hline
\end{array}
$$

$$
\begin{aligned}
2F_4^2(n) &= (2n) + (2n) + (2n) + \ldots + (2n) + (2n) = n(2n) \\
\therefore F_4^2(n) &= \tfrac{2n^2}{2} = n^2.
\end{aligned}
$$

An interesting aspect of square numbers is that one can decompose a square number into the sum of two consecutive triangular numbers. This fact can be demonstrated by simply drawing lines to connect the units of a square figurate into two right triangles, or, alternatively, it may be proven using algebra, as follows:

Theorem 2: Square numbers decompose into two consecutive triangular numbers.

Proof: Let $F_3^2(n)$ *denote an arbitrary triangular number with the previous triangular number being* $F_3^2(n-1)$. *Straight-forward computation yields the following:*

$$F_4^2(n) = n^2 = \frac{2n^2}{2}$$

$$= \frac{n^2+n}{2} + \frac{n^2-n}{2}$$

$$= \frac{n(n+1)}{2} + \frac{(n)(n-1)}{2}$$

$$= F_3^2(n) + F_3^2(n-1).$$

Hence, every square number can be expressed as the sum of two consecutive triangular numbers.

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In general, one may continue to perform this line of inquiry in order to establish the arithmetic progressions of the infinitely many figurate numbers for side count $n > 4$; however, the author now moves to establish a general formula that generates all two-dimensional figurate number formulas. First, consider the following table of figurate numbers, Table 1:

Table 1. Table of figurate numbers

Figure	Formula	Difference
Triangle	$F_3^2(n) = \frac{1n^2 - (-1)n}{2}$	$F_4^2(n) - F_3^2(n) = \frac{n^2 - n}{2}$
Square	$F_4^2(n) = \frac{2n^2 - (0)n}{2}$	$F_5^2(n) - F_4^2(n) = \frac{n^2 - n}{2}$
Pentagon	$F_5^2(n) = \frac{3n^2 - (1)n}{2}$	$F_6^2(n) - F_5^2(n) = \frac{n^2 - n}{2}$
Hexagon	$F_6^2(n) = \frac{4n^2 - (2)n}{2}$	$F_7^2(n) - F_6^2(n) = \frac{n^2 - n}{2}$
⋮	⋮	⋮

Notice that the difference between every two consecutive formulas is the same value (constant). This information is valuable in that it can be used to construct a proof using the principle of mathematical induction.

Theorem 3, the base formula for two-dimensional figurate number series

Proof: Proceeding via the principle of mathematical induction, let P(m) denote the statement that "every arithmetic series for figurate numbers may be generated by the formula $\frac{(m-2)n^2-(m-4)n}{2}$*, where* $m, n \in \mathbb{N}$ *and* $m \geq 3$*." The basis step* $m = 3$ *produces* $P(3) = \frac{n^2+n}{2}$*, which is the formula for triangular numbers and thus a true statement. Proceeding now to the inductive hypothesis, let* $P(k)$ *denote the statement that every arithmetic series for figurate numbers can be* *generated by the formula* $\frac{(k-2)n^2-(k-4)n}{2}$, *where* $k, n \in \mathbb{N}$. *The ensuing steps demonstrate that* $P(k+1)$ *necessarily follows:*

$$P(k+1) = \frac{[(k+1)-2]n^2 - [(k+1)-4]n}{2}$$

$$= \frac{(k-1)n^2 - (k-3)n}{2}$$

$$= \frac{(k-2)n^2 - (k-4)n}{2} + \frac{n^2 - n}{2}$$

$$= P(k) + \frac{n^2-n}{2}.$$

Given that $\frac{n^2-n}{2}$ *is the constant value between figurate number formulas, this proof has demonstrated that* $P(k+1)$ *is true. Therefore, by the principle of mathematical induction, every arithmetic series for two-dimensional figurate numbers can be generated by the formula* $\frac{(m-2)n^2-(m-4)n}{2}$, *where* $m, n \in \mathbb{N}$ *and* $m \geq 3$.

The Pythagoreans produced a lasting impact on mathematics that was spawned by their philosophy, which had the worship of numbers as its central tenet. The Pythagoreans' ideas influenced the works of Plato, Aristotle, and most notably Euclid, although the geometric representation had been drastically reduced from *n*-gonal numbers to only square and oblong (rectangular) numbers. Evidence of this may be found in Euclid's *The Elements*, where definitions 15-19 in Book VII discuss integers in relation to geometric orientation.[4] However, Euclid had trouble absorbing the arithmetic series of figurate numbers,[4] as noted by Heath in his translation and commentary:

> The words *plane* and *solid* applied to numbers are of course adapted from their use with reference to geometrical figures...Iamblicus tells us that in the old days they represented the quantuplicities of number in a more natural way by splitting them up into units, and not, as in our day, by symbols. Aristotle too mentions one Eurytus as having settled what number belonged to what, such a number to a man, such a number to a horse, and so on, "copying their shape with pebbles, just as those do who arrange numbers in the forms of triangles or squares."[5]

However, mathematicians who appear immediately after Euclid, such as Nichomachus, Theon of Smyrna, and Diophantus, called into question Euclid's notion of representing numbers as lines, areas, and volumes, specifically when it came to the representation of the operation "a number multiplied by one."[5] Here, Euclid's definitions encounter problems as some operations call for a construction in x dimensions yet the result appears to be in y dimensions. For example, consider the statement "three multiplied by one equals three." Because the operation is multiplication, Euclid's definitions characterize the result as a two-dimensional object, line multiplied by line, producing an area. However, the number three prior to the operation was already a one-dimensional representation (a line). Here, a line multiplied by a line did not produce an area but another line, which is paradoxical according to the working definitions. This was just one of several areas that was revised by the Neo-Pythagoreans, who kept the mathematics developed by the original Pythagorean school but chose to rework the taxonomy and philosophical components.

The Neo-Pythagoreans

One of the earliest figures in Neo-Pythagoreanism was Nicomachus, about whom little is known. Nichomachus, pictured in Figure 4, was heavily influenced by Plato and the original Pythagoreans, whose traditions were still extant in Nichomachus's day. Although Nichomachus subscribed to the many established doctrines of the Pythagoreans, he differed with their traditional views on education. Nichomachus maintained that, of the four subjects stressed by Plato's academy, arithmetics was superior.[3] The emphasis placed on arithmetics by Nicomachus was a bold assertion since the traditional model called for an equal balance of study among the various subjects. Subsequently, Nichomachus made many contributions in arithmetic and figurate numbers, where he expanded on the original ideas and invented several more.

One of Nichomachus's assertions for figurate numbers was that the "(n-1)st triangular number added to the nth k-gonal number gives the nth (k+1)-gonal number."[3] Here is a proof of Nichomachus's assertion:

Figure 4: A depiction of Nicomachus[6]

Theorem 4, Nicomachus's first figurate theorem: The nth (k+1)-gonal number may be written as the sum of the (n-1)th triangular number plus the nth k-gonal number.

Proof: Let $F_{k+1}^2(n)$ *represent the nth (k+1)-gonal number and* $F_k^2(n)$ *represent the nth k-gonal number. Applying Theorem 3 yields the following:*

$$F_{k+1}^2(n) = F_3^2(n-1) + F_k^2(n)$$

$$\frac{(k-1)n^2-(k-3)n}{2}$$

$$= \frac{(3-2)(n-1)^2-(3-4)(n-1)}{2} + \frac{(k-2)n^2-(k-4)n}{2}$$

$$= \frac{(n-1)^2+(n-1)}{2} + \frac{(k-2)n^2-(k-4)n}{2}$$

$$= \frac{n^2-n}{2} + \frac{(k-2)n^2-(k-4)n}{2}$$

$$= \frac{(k-1)n^2-(k-3)n}{2}.$$

In addition to discovering a connection among all the figurate numbers, Nichomachus made the curious discovery that the sum of the first *n* number of odd integers is equal to *n* squared (see Table 2):

Table 2. Sum of first *n* number of odd integers equals *n* squared.

The First *n* Odd Integers	Summation
1	1^2
1+3	2^2
1+3+5	3^2
1+3+5+7	4^2
1+3+5+7+9	5^2
$\vdots$	$\vdots$

He also asserted that a number cubed is equal to the summation of odd integers that start at $F_3^2(n-1)$ and run to $F_3^2(n-1)+n$ (see Table 3):.

Table 3. Summation of sequential odd integers equals *n* cubed.

Run of *n* Odd Integers	Summation
1	1^3
3+5	2^3
7+9+11	3^3
13+15+17+19	4^3
21+23+25+27+29	5^3
⋮	⋮

In honor of Nichomachus, the author offers the following proof, linking triangular numbers and square numbers (two-dimensional figurate) to cubed numbers (three-dimensional figurate).

Theorem 5, triangular numbers to cubed numbers: Every cubed integer greater than one is the difference of two square numbers whose indices are consecutive triangluar numbers.

Proof:

$$F_4^2[F_3^2(n)] - F_4^2[F_3^2(n-1)] = \left[\frac{n(n+1)}{2}\right]^2 - \left[\frac{n(n-1)}{2}\right]^2$$

$$= \frac{n^2(n+1)^2}{4} - \frac{n^2(n-1)^2}{4}$$

$$= \frac{n^4+2n^3+n^2}{4} - \frac{n^4-2n^3+n^2}{4}$$

$$= \frac{n^4+2n^3+n^2-n^4+2n^3-n^2}{4}$$

$$= \frac{4n^3}{4}$$

$$= n^3$$

The following Table 4 illustrates the results of Theorem 5:

Table 4. Triangular numbers to cubed numbers

Run of n Odd Integers	Summation
3^2-1^2	2^3
6^2-3^2	3^3
10^2-6^2	4^3
15^2-10^2	5^3
21^2-15^2	6^3
…	…

Besides Nichomachus, the other leading Neo-Pythagorean who contributed to the advancement of figurate numbers was Diophantus of Alexandria, portrayed in Figure 5. Diophantus wrote several books on mathematics and took the first steps toward the development of a completely abstract system, which later blossomed into algebra. In his work, Diophantus focused on deducing the arithmetic properties of figurate numbers, such as deducing the number of sides, the different ways a number can be expressed as a figurate number, and the formulation of the arithmetic progressions.[7]

Figure 5: A depiction of Diophantus of Alexandria[8]

It appears that Diophantus was influenced by Euclid's work but strove to re-establish Euclid's views on numbers through the syncretism of ancient Pythagorean representation. By utilizing discrete values in the form of pebbles rather than lines, Diophantus assisted in the taxonomy of numbers at the expense of neglecting the development of incommensurate numbers. Regardless of his motivation, Diophantus not only revived but also furthered the theory of figurate numbers, with one specific addition being the following algorithm, which tests whether an integer is of a certain m-gonality.[9]:

Definition: The Diophantus Algorithm

Step 1. Test an arbitrary integer n *to determine if it is a perfect square:*

$$8(m-2)F_m^2(n)+(m-4)^2=x, \quad x \in \mathbb{N}$$

Step 2. If x *is a perfect square, then* n *is obtained for the m-gonal number through*

$$2n(m-2)-(m-4).$$

The Diophantus Algorithm requires two computational steps, which is justified given the complete lack of algebra during the 2nd century CE. An alternative method to test m-gonality uses the established facts that every number can be expressed as the difference of two triangular numbers (Theorem 1) and every square number is the difference of two consecutive triangular numbers (Theorem 2).

Theorem 6, *M*-gonal test via the linear combination of consecutive triangular numbers: Every integer can be tested for m-gonality by the formula $\mathbf{x} = \mathbf{F_3^2(n) + (m-3)F_3^2(n-1), x \in \mathbb{N}}$ **.**

Proof: From Theorem 3, every two-dimensional figurate number can be expressed in terms of the formula $F_m^2(n) = \frac{(m-2)n^2-(m-4)n}{2}$. *Thus,*

$$\frac{(m-2)n^2-(m-4)n}{2} = \frac{(m-2)n^2-(m-4)n}{2} + \frac{n(n+1)}{2} - \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2} + \frac{(m-3)(n^2-n)}{2}$$

$$= \frac{n(n+1)}{2} + (m-3)\frac{n(n-1)}{2}$$

$$= F_3^2(n) + (m-3)F_3^2(n-1).$$

Therefore, every integer can be tested for m-gonality via the formula $x = F_3^2(n) + (m-3)F_3^2(n-1)$.

Theorem 6 allows the mathmatician to input an unknown value x and test arbitrary values for m where solutions have integer values for n. However, the true power of this theorem lies in the following corollary, which states that m-gonality can be checked through the well-known quadratic formula:

Corollary 1: *M*-gonal test via the quadratic formula

$$x = \frac{n^2 + n}{2} + \frac{(m-3)(n^2 - n)}{2} \Longrightarrow$$

$$(m-2)n^2 + (4-m)n - 2x = 0, \qquad A = (m-2), B = (4-m), C = -2x.$$

For example, one can test an arbitrary integer, say 215, to see if it is a hexagonal number $(m = 6)$:

$$(6-2)n^2 + (4-6)n - 2(215) = 0$$

$$4n^2 - 2n - 430 = 0$$

$$n = \frac{1 \pm \sqrt{1721}}{4}.$$

Since the desired solutions for n are positive integers, one can conclude that 215 is not a hexagonal number because these two solutions are both irrational. Alternatively, consider the number 1551 to see if it is a triagontagonal number (a 30-sided figure):

$$(30-2)n^2 + (4-30)n - 2(1551) = 0$$

$$28n^2 - 26n - 3102 = 0$$

$$n = \frac{-141}{14} \; or \; n = 11.$$

Thus, 1551 is the 11th triagontagonal number because this equation results in an integer solution.

Diophantus was one of the first mathematicians to systematically seek solutions for equations of specific forms, especially those that have multiple unknowns. For example, Diophantus commonly examined problems of the form $x^2 + y^2 = z^2$, which generates Pythagorean triplets when $x, y, z \in \mathbb{Z}$. Due to the nature of mathematics during his day, Diophantus often sought only integer solutions to these types of problems and was often satisfied upon finding a single solution. However, solving equations of this type often leads to parametric solutions where infinitely many solutions satisify the equation. In honor of Diophantus, equations of the form $Ax^n + By^n = Cz^n$ are now known as Diophantine equations. Continuing with the example of the second-degree Diophantine equation,

$x^2 + y^2 = z^2$, one sees that both $(3,4,5)$ and $(5,12,13)$ are two of the infinitely many solutions given by the parametric solutions $x = st,\ y = \frac{s^2-t^2}{2},\ z = \frac{s^2+t^2}{2}$.

The ability to solve these types of equations allows the mathemetician to answer more advanced questions on figurate numbers, such as "which triangular numbers are also square numbers?" One obvious approach would be to set up an equation in which the triangular number formula is equal to the square number formula, resulting in the Diophantine equation (alternatively called a Pell equation since there is only one coefficient),

$$\frac{s^2+s}{2} = t^2n^2 + n = 2m^2 .$$

As mentioned earlier with parametric solutions, mathematicians of Diophantus's time were satisfied with a single, non-trivial solution like (8,6): $(8)^2 + (8) = 2(6)^2 = 72$. These solutions were often obtained through trial and error, and it was not until 1778 that the great Leonhard Euler showed that a parametric solution existed for which triangular numbers are also square numbers:

$$N_k = \left[\frac{\left(3+2\sqrt{2}\right)^k - \left(3-2\sqrt{2}\right)^k}{4\sqrt{2}}\right]^2 .$$ [10]

After Diophantus, a large shift occurred in the realm of mathematics, and development was neglected for several centuries until Fibonnaci reintroduced the western world to mathematics in the 12th century CE. Building on Fibonacci and Diophantus, the next major steps were taken by the pioneering fathers of modern number theory, Blaise Pascal and Pierre de Fermat.

Birth of Modern Number Theory

During the 17th century, mathematics underwent sweeping reforms in the western world. A large majority of these revolutionary skirmishes took place in response to the mathematicians Descartes, Pascal, and Fermat, whose work in arithmetic and algebraic methods challenged the prevailing models during their time.[11,12] Several of these newer developments resulted in famous mathematical feuds, as the new creative methods were demonstrating the same explanatory power as geometry without the need for geometric principles.

Blaise Pascal (see Figure 6) excelled at mathematics and science even as a child. In his early adulthood, he created a precursor to the calculator known as Pascal's calculator, or a Pascaline. Pascal's work in mathematics spanned several different areas, with his most recognized contribution being his work in probability theory. In 1653, Pascal wrote his *Treatise on the Arithmetic Triangle,* which discusses the triangular array known as Pascal's triangle. This triangle—rife with numerical patterns and mathematical connections—is primarily used for the identification of

Figure 6: Blaise Pascal[13]

binomial coefficients. As shown in Figure 7, one of the more interesting patterns that occurs in Pascal's triangle is along the third diagonal in both directions: the reader's old friends, the triangular numbers. In fact, all of the rows for Pascal's triangle correspond to figurate numbers, known as the figurate numbers of simplices. In geometry, a simplex is the consideration of a triangle in arbitrary dimensions, and, in the notation used here, the simplex is every arithmetic series of the form $F_3^x(n)$ where $x \in \mathbb{N}$. Pascal was not the first person to establish this triangular array, as much work on the figurate numbers of simplices had already been carried out by Chinese and Neo-Pythagorean mathematicians in previous centuries. However, Pascal was instrumental in making the previously unidentified connections between binomial coefficients and combinatorics.[11]

1

1 **1**

1 **2** **1**

1 **3** **3** **1**

1 **4** **6** **4** **1**

1 **5** **10** **10** **5** **1**

.. ..

Figure 7: Pascal's Triangle (triangular numbers are shaded)

Offered in the spirit of Pascal, the following proof demonstrates that every positive consecutive integer product is divisible by the summation of integers less n.

Theorem 7: The positive consecutive summation of n-1 terms divides a positive consecutive product of n terms. The product of positive consecutive integers $\mathbf{n}!$ is divisible by the summation less n: $\mathbf{F_3^2(n-1) \mid n!}$.

Proof:

$$\frac{n!}{F_3^2(n-1)} = \frac{1*2*3*...*(n-1)*n}{1+2+3+\cdots(n-2)+(n-1)}$$

$$= \frac{1*2*3*...*(n-1)*n}{\frac{(n-1)*(n-2)}{2}}$$

$$= \frac{2*2*3*\ldots*(n-1)*n}{(n-2)*(n-1)}$$

$$= 2*2*3*\ldots*(n-4)*(n-3)*n.$$

Corollary 2: Every triangular number greater than three is composite.

In Pierre de Fermat's time, algebra had matured to the point that it was powerful enough to challenge the notion that geometry was the bedrock of mathematics, and it was these advancements that enabled Fermat to develop significant components of the theoretical nature of numbers. It must be understood that during the Age of Reason, the profession of mathematician was non-existent; hence, for Pascal, Fermat, Descartes and company, mathematics was seen more as a leisure activity than an academic pursuit: most of the mathematical advancements during this time were made via public intellectual contests and personal correspondence between academicians. Pierre de Fermat (see Figure 8) was no exception to this rule, and the bulk of his contributions are known through Fermat's correspondence with friends and colleagues.[14]

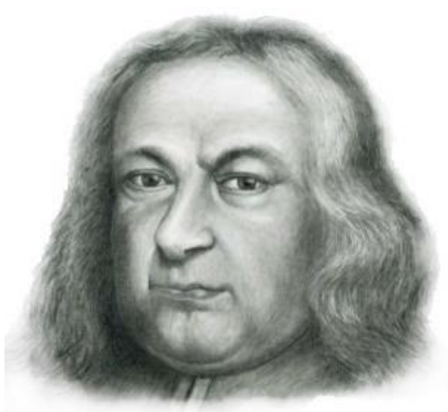

Figure 8. Pierre de Fermat[15]

Besides the tomes of personal correspondence left by Fermat, some of his work comes from notes scribbled in his personal books. Fermat penned one of his most legendary passages in the margin of his copy of Diophantus's *Arithmeticae*: "I have discovered a truly remarkable proof of this theorem which this margin is too small too contain."[14] In similar vein, many of Fermat's contributions to mathematics appear as motivational challenges to the math community at large. In regard to his work on figurate numbers, Fermat pushed Diophantus's work further by creating even more connections. One of the most famous of these was Fermat's figurate number theorem, which states that every number n can be written as the sum of at most n figurate numbers of n-gonal sides. While it seems unlikely that Fermat ever solved this conjecture due to the lack of required mathematical theory in his time, several advances were made over the next century and the eventual proof was derived in 1813 by Cauchy.

After Fermat, number theory and figurate numbers were never the same: new ideas were formulated and several gaps were bridged. For instance, mathematicians began noticing connections such as the relationship of triangular numbers to Pell-Lucas numbers (fractions whose values increasingly approximate $1+\sqrt{2}$, the silver ratio).

Theorem 8, deriving non-trivial factors of triangular numbers from Pell-Lucas numbers

Proof: It is well-known that $(P_n + P_{n-1})^2(P_n)^2 = F_3^2(n)$ *, where* P_n, P_{n-1} *denote Pell-Lucas numbers. Since every triangular number greater than three is composite (Corollary 2), one seeks to establish possible non-trivial factor pairs:*

$$\frac{n(n+1)}{2} = \frac{4}{4}\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{4n^2+4n}{8}$$

$$= \frac{(2n+1)^2-(-1)^{2n}}{8}$$

$$= \frac{(2n+1)+(-1)^n}{2} * \frac{(2n+1)-(-1)^n}{4}.$$

From the Pell identity it is known tha t $(P_n + P_{n-1})^2 = \frac{2n+1+(-1)^n}{2}$ *and* $P_n^2 = \frac{2n+1-(-1)^n}{4}$ *because* $P_n \leq P_n + P_{n-1}$ *. Now, since the term* $(-1)^n$ *alternates signs as n alternates, there are two cases for each equation, making a total of four required solutions. Arbitrarily choosing to solve* $(P_n + P_{n-1})^2$ *first, the mathematician finds the following:*

<u>Case 1: n is even</u>

$$(P_n + P_{n-1})^2 = \frac{2n+2}{2}$$

$$= n+1.$$

<u>Case 2: n is odd</u>

$$(P_n + P_{n-1})^2 = \frac{2n}{2}$$

$$= n.$$

And now the mathematician solves P_k^2 *for solutions:*

<u>Case 3: n is even</u>

$$(P_n)^2 = \frac{2n}{4}$$

$$= \frac{n}{2}.$$

<u>Case 4: n is odd</u>

$$(P_k)^2 = \frac{2n+2}{4}$$

$$= \frac{n+1}{2}.$$

In conclusion, because one seeks factors of triangular numbers, it follows that when n *is even, case 1 and case 3 are the factors. Similarly, when* n *is odd, case 2 and case 4 are the factors.*

Figurate Numbers in Modern Times

In the present day, the rise of science has shifted the focus for mathematicians, and developments in figurate numbers have mostly stalled. Mathematics with immediate connections to physical applications, such as calculus and differential equations, have taken priority over the more philosophical theories, which have become novelties. For figurate numbers, the advancements have largely been grounded in geometric properties, such as mirroring the different polytopes and creating new arithmetic series corresponding to figures in higher dimensions. For example, the following Table 5 outlines the expansion of triangular numbers into the third and fourth dimensions:

Table 5: Figurate numbers in higher dimensions

geometric object	dimension	n=1	n=2	n=3	n=4	n=5	...
triangle	2	1	3	6	10	15	...
tetrahedron	3	1	4	10	20	35	...
hypertetrahedron	4	1	5	15	35	70	...

Similar to the expansion in higher dimensions, figurate numbers continue to be developed in two dimensions under alternative discrete arrangements. For example, Table 6 compares hexagonal numbers, which form the traditional figurate number set, with their cousins, the centered hexagonal numbers:

Table 6: An alternative figurate number arrangement

geometric object	formula	n=1	n=2	n=3	n=4	n=5	…
hexagonal	$2n^2 - n$	1	6	15	28	45	...
centered hexagonal	$n^3 - (n-1)^3$	1	7	19	37	61	...

Unfortunately, the continuing advancements made in the field of figurate numbers have not been sufficient to garner the attention of serious mathematicians, who have largely shifted their focus to other theories that have immediate application. This emphasis has been brought about by the scientific revolution, which continues to guide the role and purpose of mathematics in contemporary societies. However, even though the theory of figurate numbers does not have any apparent modern use, history has shown that previously established mathematics sometimes have an unforseen purpose.

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